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Multivariate Padé Approximants Associated with Functional Relations

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We explicitly construct non-homogeneous multivariate Padé approximants to some functions like

$$G(x, y) := \sum_{k=0}^{\infty} q^{-k} \prod_{j=0}^{k} (1 + q^{-j}x + q^{-2j}xy),$$

$$F(x, y) := \sum_{i, j=0}^{\infty} q^{-(i+j)^2/2} x^i y^j,$$

and

$$E(x, y) := \sum_{i, j=0}^{\infty} \frac{x^i y^j}{[i+j]_q !},$$

which satisfy some functional equations, where $|q| > 1$, $q \in \mathbb{C}$.

1. INTRODUCTION

By using the residue theorem and the functional equation method, Prof. P. B. Borwein [1] has successfully constructed one variable Padé approximants to the q-elementary functions like q analogues of exp, log, and partial theta functions. Similarly in [8], we have constructed non-homogeneous multivariate Padé approximants to some functions which satisfy a simple functional equation

$$F(q^{\mu} x, q^{\nu} y) = R_1(x, y)\, F(x, y), \qquad q \in \mathbb{C}, \tag{1.1}$$

where $F(x, y)$ is an entire function in $\mathbb{C} \times \mathbb{C}$, $R_1(x, y)$ is a polynomial of degree 1 in x and y, and μ, ν are non-negative integers. As the situation is much more complicated in the multivariate case than in the univariate

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case, only a few functions satisfy the above functional equation (1.1). Our intention in this paper is to show how to construct non-homogeneous multivariate Padé approximants, especially the denominators of these approximants, to some functions which satisfy some comparatively complicated functional equations. In order to avoid notational difficulties, we will restrict ourselves to the case of bivariate functions. The generalization to more than two variables is straightforward (see [4]).

We believe the results to be of interest because there are very few explicit examples of multivariate Padé approximants. It is noteworthy that much of the difficulty in finding explicit formulae for multivariate Padé approximants lies in the determination of appropriate index sets for the numerator and denominator polynomials. It is in the search for the latter that much of the work of the paper lay.

We recall

DEFINITION. Let

$$F(x, y) := \sum_{(i,\, j) \in \mathbb{N}^2} c_{ij} x^i y^j, \qquad c_{ij} \in \mathbb{C} \tag{1.2}$$

be a formal power series, and let M, N, E be index sets in $\mathbb{N} \times \mathbb{N} =: \mathbb{N}^2$. The *$(M, N)$ non-homogeneous multivariate Padé approximant to $F(x, y)$ on the finite set E* is a rational function

$$[M/N]_E (x, y) := \frac{P(x, y)}{Q(x, y)} \tag{1.3}$$

with polynomials

$$P(x, y) := \sum_{(i,\, j) \in M} a_{ij} x^i y^j, \qquad a_{ij} \in \mathbb{C}, \tag{1.4}$$

$$Q(x, y) := \sum_{(i,\, j) \in N} b_{ij} x^i y^j, \qquad b_{ij} \in \mathbb{C}, \tag{1.5}$$

and an interpolation set E, such that

$$(FQ - P)(x, y) = \sum_{(i,\, j) \in \mathbb{N}^2 \setminus E} d_{ij} x^i y^j, \qquad d_{ij} \in \mathbb{C} \tag{1.6}$$

with

$$M \subseteq E, \tag{1.7}$$

$$\#(E \setminus M) = \# N - 1 \tag{1.8}$$

and E satisfies the inclusion property:

$$(i, j) \in E, \qquad 0 \leqslant k \leqslant i, \qquad 0 \leqslant l \leqslant j \Rightarrow (k, l) \in E. \tag{1.9}$$

The reader may find properties of non-homogeneous multivariate Padé approximants discussed in Cuyt [2, 3].

We also need the standard q analogues of factorials and binomial coefficients. The *q-factorial* is

$$[n]_q! := [n]! := \frac{(1-q^n)(1-q^{n-1}) \cdots (1-q)}{(1-q)^n}, \tag{1.10}$$

where $[0]_q! := 1$. Since $(1-q^n)/(1-q) = 1 + q + \cdots + q^{n-1}$, it is clear that

$$\lim_{q \to 1} [n]_q! = n!. \tag{1.11}$$

The *q-binomial coefficient* is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]! \cdot [n-k]!}, \tag{1.12}$$

and as above

$$\lim_{q \to 1} \begin{bmatrix} n \\ j \end{bmatrix}_q = \binom{n}{k}. \tag{1.13}$$

We note that

$$\prod_{\substack{h=0 \\ h \neq k}}^{n} (q^k - q^h) = (-1)^k \, q^{k(2n-k-1)/2} [n-k]! \, [k]! \, (1-q)^n. \tag{1.14}$$

We state our results and give examples in Section 2, and prove all the results in Section 3.

2. RESULTS AND EXAMPLES

Our first main result deals with an interpolation set E that is triangular in nature; the index set N defining the denominator is square, while the index set M defining the numerator is the union of a triangle and a square; in the case $m \geqslant 2n$, it reduces to a triangle. In the proof of Theorem 2.1, we present diagrams of the interpolation, numerator, and denominator index sets.

THEOREM 2.1. *Let* $m, n \in \mathbb{N}$, $m \geqslant n+1 \geqslant 1$, *and*

$$W := \{(i, j)\colon 0 \leqslant i+j \leqslant m,\ i, j \geqslant 0\}; \tag{2.1}$$

$$N := \{(i, j)\colon 0 \leqslant i, j \leqslant n\}; \tag{2.2}$$

$$M := N \cup W; \tag{2.3}$$

$$E := \{(i, j)\colon 0 \leqslant i+j \leqslant m+n,\ i, j \geqslant 0\}. \tag{2.4}$$

Let $q \in \mathbb{C}$ *and* $F(x, y)$ *be entire in* $\mathbb{C} \times \mathbb{C}$ *and satisfy the functional equation*

$$F(qx, qy) = R_1(x, y)\, F(x, y) + R_0(x, y), \tag{2.5}$$

where $R_1(x, y)$ *is a polynomial of degree* 1 *in* x *and* y, *and* $R_0(x, y)$ *is a polynomial of degree at most* 1 *in* x *and* y. *Let*

$$I(x, y) := \frac{1}{2\pi i} \int_\Gamma \frac{F(tx, ty)\, dt}{\left(\prod_{k=0}^{n} (t-q^k)\right) t^{m+1}}, \tag{2.6}$$

where Γ *is a circular contour containing* $0, q^0, q^1, ..., q^n$, *and let*

$$W_k(w) := \frac{(-1)^k}{(1-q)^n\, [n]!} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-1)/2 - k(m+n)} \tag{2.7}$$

$$A(x, y) := \sum_{k=0}^{n} W_k(q) \prod_{j=0}^{k-1} R_1(q^j x, q^j y), \tag{2.8}$$

$$\begin{aligned} B(x, y) := {} & \sum_{k=0}^{n} W_k(q) \sum_{j=0}^{k-1} R_0(q^j x, q^j y) \prod_{i=j+1}^{k-1} R_1(q^i x, q^i y) \\ & + \frac{1}{m!} \frac{d^m}{dt^m} \left\{ \frac{F(tx, ty)}{\prod_{k=0}^{n} (t-q^k} \right\}_{t=0}. \end{aligned} \tag{2.9}$$

Then

(i) $$I(x, y) = A(x, y)\, F(x, y) + B(x, y); \tag{2.10}$$

(ii) $$I(x, y) = \sum_{(i, j) \in \mathbb{N}^2 \setminus E} d_{ij} x^i y^j, \qquad d_{ij} \in \mathbb{C}; \tag{2.11}$$

(iii) $$A(x, y) = \sum_{(i, j) \in N} a_{ij} x^i y^j, \qquad a_{ij} \in \mathbb{C}; \tag{2.12}$$

$$B(x, y) = \sum_{(i, j) \in M} b_{ij} x^i y^j, \qquad b_{ij} \in \mathbb{C}; \tag{2.13}$$

(iv) $$M \subseteq E \qquad \text{and} \qquad \#(E \setminus M) \geqslant \#N - 1, \tag{2.14}$$

and then the (M, N) *non-homogeneous multivariate Padé approximant to* $F(x, y)$ *on set* E *is*

$$[M/N]_E(x, y) = -\frac{B(x, y)}{A(x, y)}.$$

Remark. Here if we let

$$F(x, y) := \sum_{i,\, j=0}^{\infty} c_{ij} x^i y^j, \qquad c_{ij} \in \mathbb{C},$$

then $B(x, y)$ can be expressed as

$$\begin{aligned} B(x, y) := &\sum_{k=0}^{n} W_k(q) \sum_{j=0}^{k-1} R_0(q^j x, q^j y) \prod_{i=j+1}^{k-1} R_1(q^i x, q^i y) \\ &+ \frac{(-1)^{n+1}}{q^{n(n+1)/2}} \sum_{l=0}^{m} \left(\sum_{\substack{j_0, \ldots, j_n > 0 \\ j_0 + \cdots + j_n = m-l}} q^{-\sum_{k=0}^{n} k j_k} \right) \sum_{\substack{r, s > 0 \\ r+s=l}} c_{rs} x^r y^s. \end{aligned} \tag{2.15}$$

EXAMPLE 1. Let $q \in \mathbb{C}$, $|q| > 1$, and

$$G(x, y) := \sum_{k=0}^{\infty} q^{-k} \prod_{j=0}^{k} (1 + q^{-j} x + q^{-2j} xy). \tag{2.16}$$

Then the functional relation for $G(x, y)$ is

$$\begin{aligned} G(qx, qy) &= \sum_{k=0}^{\infty} q^{-k} \prod_{j=0}^{k} (1 + q^{-j+1} x + q^{-2j+2} xy) \\ &= (1 + qx + q^2 xy) + \sum_{k=1}^{\infty} q^{-k} \prod_{j=0}^{k} (1 + q^{-j+1} x + q^{-2j+2} xy) \\ &= (1 + qx + q^2 xy) + \sum_{k=0}^{\infty} q^{-k-1} \prod_{j=0}^{k+1} (1 + q^{-j+1} x + q^{-2j+2} xy) \\ &= (1 + qx + q^2 xy) + q^{-1} \sum_{k=0}^{\infty} (1 + qx + q^2 xy)\, q^{-k} \\ &\quad \times \prod_{j=0}^{k} (1 + q^{-j} x + q^{-2j} xy) \\ &= (1 + qx + q^2 xy) + q^{-1}(1 + qx + q^2 xy)\, G(x, y). \end{aligned}$$

So

$$R_1(x, y) = q^{-1}(1 + qx + q^2 xy); \qquad R_0(x, y) = 1 + qx + q^2 xy.$$

Now let N, M, E be defined as in (2.2), (2.3), (2.4), respectively, and let

$$Q(x, y) := \sum_{k=0}^{n} W_k(q)\, q^{-k} \prod_{j=1}^{k} (1 + q^j x + s^{2j} xy), \tag{2.17}$$

$$P(x, y) := \sum_{k=0}^{n} W_k(q) \sum_{j=1}^{k} q^{j-k} \prod_{i=j+1}^{k} (1 + q^i x + q^{2i} xy) + \frac{1}{m!} \frac{d^m}{dt^m} \left\{ \frac{G(tx, ty)}{\prod_{k=0}^{n} (t - q^k)} \right\}_{t=0}, \tag{2.18}$$

where $W_k(q)$ is defined by (2.7). Then by the theorem above, we have for $G(x, y)$

$$[M/N]_E (x, y) = -\frac{P(x, y)}{Q(x, y)}. \tag{2.19}$$

As a preliminary to our second main result, we state some simple consequences of a functional relation.

THEOREM 2.2. *Let $F(x, y)$ be entire in $\mathbb{C} \times \mathbb{C}$ and satisfy the functional equation*

$$F(qx, qy) = R_1(x, y)\, F(x, y) + u(x)\, F(x, 0) + v(y)\, F(0, y) + R_0(x, y), \tag{2.20}$$

where $R_1(x, y)$ is a polynomial of degree 1 in x and y, $u(x)$ and $v(y)$ are polynomials of degree at most 1 in x and y, respectively, and $R_0(x, y)$ is a polynomial of degree at most 1 in x and y.

(a) *Then*

$$F(qx, 0) = (R_1(x, 0) + u(x))\, F(x, 0) + v(0)\, F(0, 0) + R_0(x, 0) =: u_1(x)\, F(x, 0) + u_0(x); \tag{2.21}$$

$$F(0, qy) = (R_1(0, y) + v(y))\, F(0, y) + u(0)\, F(0, 0) + R_0(0, y) =: v_1(y)\, F(0, y) + v_0(y). \tag{2.22}$$

(b) *For integers* $k \geqslant 1$,

$$F(q^k x, 0) = \left(\prod_{j=0}^{k-1} u_1(q^j x)\right) F(x, 0) + \sum_{j=0}^{k-1} u_0(q^j x) \prod_{i=j+1}^{k-1} u_1(q^i x)$$

$$=: u_{1,k}(x)\, F(x, 0) + u_{0,k}(x); \tag{2.23}$$

$$F(0, q^k y) = \left(\prod_{j=0}^{k-1} v_1(q^j y)\right) F(0, y) + \sum_{j=0}^{k-1} v_0(q^j y) \prod_{i=j+1}^{k-1} u_1(q^i y)$$

$$=: v_{1,k}(y)\, F(0, y) + v_{0,k}(y). \tag{2.24}$$

(c) *For integers* $k \geqslant 1$,

$$F(q^k x, q^k y) = R_{1,k}(x, y)\, F(x, y) + S_k(x, y)\, F(x, 0) + T_k(x, y)\, F(0, y) + R_{0,k}(x, y), \tag{2.25}$$

where

$$R_{1,k}(x, y) := \sum_{j=0}^{k-1} R_1(q^j x, q^j y), \tag{2.26}$$

$$S_k(x, y) := \sum_{j=0}^{k-1} u(q^j x)\, u_{1,j}(x) \prod_{i=j+1}^{k-1} R_1(q^i x, q^i y), \tag{2.27}$$

$$T_k(x, y) := \sum_{j=0}^{k-1} v(q^j y)\, v_{1,j}(y) \prod_{i=j+1}^{k-1} R_1(q^i x, q^i y), \tag{2.28}$$

and

$$R_{0,k}(x, y) := \sum_{j=0}^{k-1} \left[R_0(q^j x, q^j y) + u(q^j x)\, u_{0,j}(x) + v(q^j y)\, v_{0,j}(y)\right] \times \prod_{i=j+1}^{k-1} R_1(q^i x, q^i y). \tag{2.29}$$

(*Here we set* $v_{1,0} = u_{1,0} := 1$; $v_{0,0} = u_{0,0} := 0$.)

Now we can give explicit expressions for multivariate Padé approximants to functions satisfying (2.20). Again our index set N for the denominator polynomial is a square, while the index set M is a union of a triangle and two rectangles; and the interpolation set E is a union of a triangle and a trapezium.

THEOREM 2.3. *Let* $m, n \in \mathbb{N}$, $m \geqslant (n+3)/2$, $n \geqslant 0$, *and*

$$U := \{(i, j): 0 \leqslant i \leqslant m+2n, 0 \leqslant j \leqslant n\}; \tag{2.30}$$

$$V := \{(i, j): 0 \leqslant i \leqslant n, 0 \leqslant j \leqslant m+2n\}; \tag{2.31}$$

$$W := \{(i, j): 0 \leqslant i+j \leqslant m+2n, i, j \geqslant 0\}; \tag{2.32}$$

$$N := \{(i, j): 0 \leqslant i, j \leqslant n\}; \tag{2.33}$$

$$M := U \cup V \cup W; \tag{2.34}$$

$$E := \{(i, j): 0 \leqslant i, j \leqslant m+2n, i+j \leqslant m+3n\}. \tag{2.35}$$

Let $F(x, y)$ *be entire in* $\mathbb{C} \times \mathbb{C}$ *and satisfy the functional equation* (2.20), *and let*

$$I(x, y) = \frac{1}{2\pi i} \int_{\Gamma} \frac{F(tx, ty)\, dt}{(\prod_{k=0}^{n} (t-q^k))\, t^{m+2n+2}}, \tag{2.36}$$

where Γ *is a circular contour containing* $0, q^0, q^1, ..., q^n$ (*see Figs.* 1–4).

Let

$$W'_k(q) := \frac{(-1)^k}{(1-q)^n\, [n]!} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-1)/2 - k(m+3n)}, \tag{2.37}$$

$$R(x, y) := \sum_{k=0}^{n} W'_k(q)\, R_{1,k}(x, y), \tag{2.38}$$

$$S(x, y) := \sum_{k=0}^{n} W'_k(q)\, S_k(x, y), \tag{2.39}$$

$$T(x, y) := \sum_{k=0}^{n} W'_k(q)\, T_k(x, y), \tag{2.40}$$

and

$$D(x, y) := \sum_{k=0}^{n} W'_k(q)\, R_{0,k}(x, y) + \frac{1}{(m+2n)!} \frac{d^{m+2n}}{dt^{m+2n}} \left\{ \frac{F(tx, ty)}{\prod_{k=0}^{n} (t-q^k} \right\}_{t=0}, \tag{2.41}$$

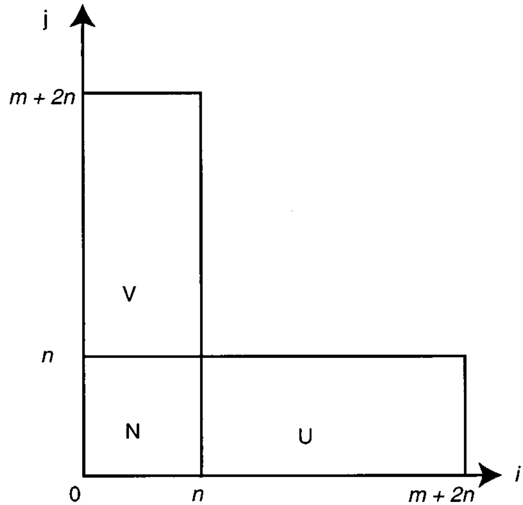

Fig 1. Index set N for the denominator polynomial.

where $R_{1,k}(x, y)$, $S_k(x, y)$, $T_k(x, y)$, *and* $R_{0,k}(x, y)$ *are defined by* (2.26) *to* (2.29), *respectively. Then*

$$\text{(i)}\quad I(x, y) := R(x, y)\, F(x, y) + S(x, y)\, F(x, 0)$$

$$T(x, y)\, F(0, y) + D(x, y); \tag{2.42}$$

$$\text{(ii)}\quad I(x, y) := \sum_{(i,j)\in \mathbb{N}^2\setminus E} d_{ij} x^i y^j, \qquad d_{ij} \in \mathbb{C}; \tag{2.43}$$

(iii) *$D(x, y)$ is a polynomial on the set W, $R(x, y)$, $S(x, y)$, and $T(x, y)$ are all polynomials on the set N, and we can set*

$$S(x, y) := \sum_{i,j=0}^{n} s_{ij} x^i y^j, \qquad s_{ij} \in \mathbb{C}, \tag{2.44}$$

$$T(x, y) := \sum_{i,j=0}^{n} t_{ij} x^i y^j, \qquad t_{ij} \in \mathbb{C}; \tag{2.45}$$

(iv) *if we let*

$$F_k(x, 0) := \sum_{i=0}^{k} c_{i0} x^i, \tag{2.46}$$

$$F_l(0, y) := \sum_{j=0}^{l} c_{0j} y^j, \tag{2.47}$$

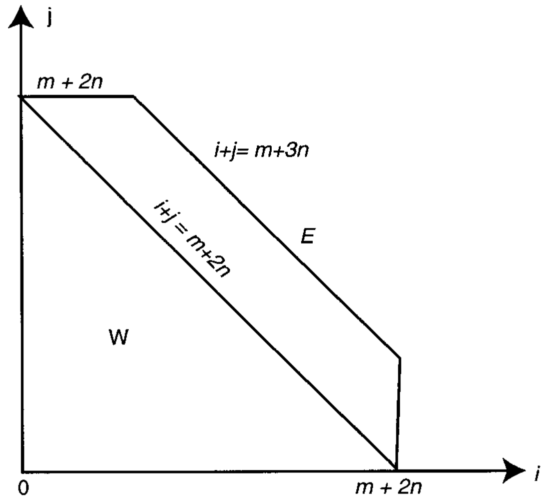

Fig. 2. Interpolation set E.

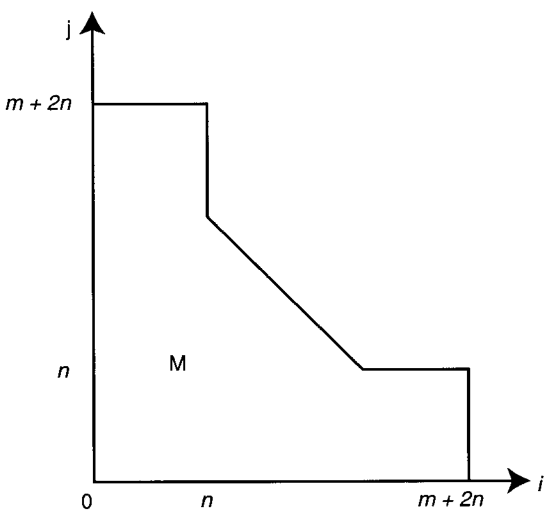

Fig. 3. Index set M for the numerator polynomial.

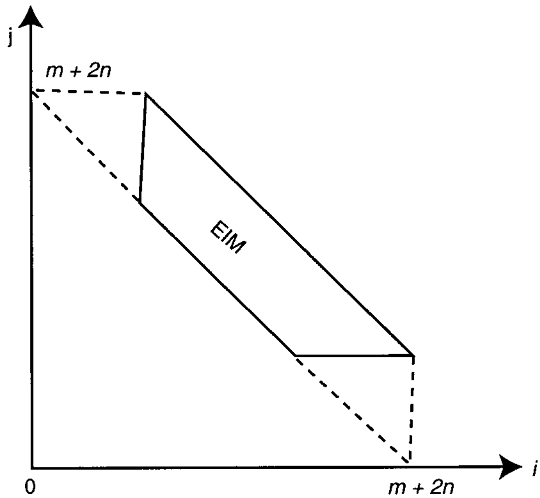

Fig. 4. Set $E\backslash M$ in conclusion (v) of Theorem 2.3.

and let

$$B(x, y) := \sum_{i,\, j=0}^{n} s_{ij} x^i y^j F_{m+2n-i}(x, 0) + \sum_{i,\, j=0}^{n} t_{ij} x^i y^j F_{m+2n-j}(0, y) + D(x, y), \tag{2.48}$$

then

$$B(x, y) = \sum_{(i,\, j) \in M} b_{ij} x^i y^j, \qquad b_{ij} \in \mathbb{C}; \tag{2.49}$$

and

$$R(x, y)\, F(x, y) + B(x, y) = \sum_{(i,\, j) \in \mathbb{N}^2 \backslash E} e_{ij} x^i y^j, \qquad e_{ij} \in \mathbb{C}; \tag{2.50}$$

$$\text{(v)} \quad M \subseteq E \qquad \textit{and} \qquad \#(E \backslash M) \geqslant \#N - 1, \tag{2.51}$$

and then the (M, N) *non-homogeneous multivariate Padé approximant to* $F(x, y)$ *on the set* E *is*

$$[M/N]_E\,(x, y) = -\frac{B(x, y)}{R(x, y)}.$$

Remark. Again if we let

$$F(x, y) := \sum_{i, j=0}^{\infty} c_{ij} x^i y^j,$$

then $D(x, y)$ defined by (2.41) can be expressed as

$$D(x, y) := \sum_{k=0}^{n} W'_k(q)\, R_{0,k}(x, y) + (-1)^{n+1} q^{-n(n+1)/2} \times \sum_{l=0}^{m+2n} \Bigg(\sum_{\substack{j_0, \ldots, j_n \geqslant o \\ j_0 + \cdots + j_n = m+2n-l}} q^{-\sum_{k=0}^{n} k j_k} \Bigg) \sum_{\substack{r, s \geqslant 0 \\ r+s=l}} c_{rs} x^r y^s,$$

and then $B(x, y)$ defined by (2.48) can be expressed as

$$\begin{aligned} B(x, y) := {} & \sum_{i, j=0}^{n} s_{ij} x^i y^j F_{m+2n-i}(x, 0) + \sum_{i, j=0}^{n} t_{ij} x^i y^j F_{m+2n-j}(0, y) \\ & + \sum_{k=0}^{n} W'_k(q)\, R_{0,k}(x, y) + (-1)^{n+1} q^{-n(n+1)/2} \\ & \times \sum_{l=0}^{m+2n} \Bigg(\sum_{\substack{j_0, \ldots, j_n \geqslant o \\ j_0 + \cdots + j_n = m+2n-l}} q^{-\sum_{k=0}^{n} k j_k} \Bigg) \sum_{\substack{r, s \geqslant 0 \\ r+s=l}} c_{rs} x^r y^s. \end{aligned} \tag{2.52}$$

Example 2. Let $|q| > 1$, and

$$F(x, y) := \sum_{i, j=0}^{\infty} q^{-(i+j)^2/2} x^i y^j. \tag{2.53}$$

This is a multivariate form of the partial theta function investigated in [7]:

$$\begin{aligned} F(qx, qy) &= \sum_{i, j=0}^{\infty} q^{-(i+j)^2/2 + (i+j)} x^i y^j \\ &= \sum_{i=0}^{\infty} q^{-i^2/2 + i} x^i + \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} q^{-(i+j)^2/2 + (i+j)} x^i y^j \\ &= 1 + \sum_{i=0}^{\infty} q^{-(i+1)(i-1)/2} x^{i+1} + \sum_{i, j=0}^{\infty} q^{-(i+j+1)(i+j-1)/2} x^i y^{j+1} \\ &= 1 + q^{1/2} x \sum_{i=0}^{\infty} q^{-i^2/2} x^i + q^{1/2} y \sum_{i, j=0}^{\infty} q^{-(i+j)^2/2} x^i y^j \\ &= 1 + q^{1/2} x F(x, 0) + q^{1/2} y F(x, y). \end{aligned} \tag{2.54}$$

Similarly,

$$F(qx, qy) = 1 + q^{1/2}yF(0, y) + q^{1/2}xF(x, y). \tag{2.55}$$

Equations (2.54) + (2.55):

$$\begin{aligned} F(qx, qy) = {} & \tfrac{1}{2}q^{1/2}(x+y)\, F(x, y) + \tfrac{1}{2}q^{1/2}xF(x, 0) \\ & + \tfrac{1}{2}q^{1/2}yF(0, y) + 1 \end{aligned} \tag{2.56}$$

so if we let

$$R_1(x, y) = \tfrac{1}{2}q^{1/2}(x+y), \tag{2.57}$$

and define M, N, E as in Theorem 2.3, then we can explicitly construct the (M, N) non-homogenous multivariate Padé approximant to $F(x, y)$ on the set E. The denominator of $[M/N]_E(x, y)$ to $F(x, y)$ is

$$\begin{aligned} R(x, y) &= \sum_{k=0}^{n} W'_k(q) \prod_{j=0}^{k-1} R_1(q^j x, q^j y) \\ &= \frac{1}{(1-q)^n\,[n]!} \sum_{k=0}^{n} \frac{(-1)^k}{2^k} \begin{bmatrix} n \\ k \end{bmatrix} q^{k^2 - k(m+3n+1/2)} (x+y)^k, \end{aligned} \tag{2.58}$$

by (2.37).

EXAMPLE 3. Let $|q| > 1$, $[n]! := [n]_q!$, and

$$E(x, y) := \sum_{i, j=0}^{\infty} \frac{x^i y^j}{[i+j]!}. \tag{2.59}$$

Then

$$\begin{aligned} E(qx, qy) &= \sum_{i, j=0}^{\infty} \frac{q^{i+j} x^i y^j}{[i+j]!} \\ &= \sum_{i, j=0}^{\infty} \frac{(q^{i+j}-1)\, x^i y^j}{[i+j]!} + \sum_{i, j=0}^{\infty} \frac{x^i y^j}{[i+j]!} \\ &= \sum_{i=0}^{\infty} \frac{(q^i-1)\, x^i}{[i]!} + \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \frac{(q^{i+j}-1)\, x^i y^j}{[i+j]\,!} + E(x, y) \\ &= \sum_{i=1}^{\infty} \frac{(q-1)\, x^i}{[i-1]!} + \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \frac{(q-1)\, x^i y^j}{[i+j-1]!} + E(x, y) \\ &= \sum_{i=0}^{\infty} \frac{(q-1)\, x^{i+1}}{[i]!} + \sum_{i, j=0}^{\infty} \frac{(q-1)\, x^i y^{j+1}}{[i+j]!} + E(x, y) \\ &= (q-1)\, xE(x, 0) + (q-1)\, yE(x, y) + E(x, y) \\ &= [1 + (q-1)\, y]\, E(x, y) + (q-1)\, xE(x, 0). \end{aligned} \tag{2.60}$$

Similarly,

$$E(qx, qy) = [1 + (q-1)\,x]\, E(x, y) + (q-1)\, yE(0, y). \tag{2.61}$$

So

$$\begin{aligned} E(qx, qy) = {} & [1 + \tfrac{1}{2}(q-1)(x+y)]\, E(x, y) \\ & + \tfrac{1}{2}(q-1)\, xE(x, 0) + \tfrac{1}{2}(q-1)\, yE(0, y). \end{aligned} \tag{2.62}$$

So if we let

$$R_1(x, y) := 1 + \tfrac{1}{2}(q-1)(x+y), \tag{2.63}$$

and define M, N, E as in Theorem 2.3, then we can explicitly construct the (M, N) non-homogeneous multivariate Padé approximant to $E(x, y)$ on the set E, and the denominator of $[M/N]_E(x, y)$ to $E(x, y)$ is

$$\begin{aligned} R(x, y) &= \sum_{k=0}^{n} W'_k(q) \prod_{j=0}^{k-1} R_1(q^j x, q^j y) \\ &= \frac{1}{(1-q)^n\, [n]!} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-1)/2 - k(m+3n)} \\ &\quad \times \prod_{j=0}^{k-1} R_1(q^j x, q^j y). \end{aligned} \tag{2.64}$$

Next we consider the same interpolation, numerator, and denominator sets as in Theorem 2.3, with a different functional equation. There is overlap between the type of functions satisfying (2.20) and those satisfying (2.66).

THEOREM 2.4. *Let U, V, W, N, M, E be defined as in Theorem 2.3. Let $q \in \mathbb{C}$, and*

$$F(x, y) := F_q(x, y) := \sum_{i, j=0}^{\infty} c_{ij} x^i y^j \tag{2.65}$$

be entire in $\mathbb{C} \times \mathbb{C}$ and let it satisfy the following functional equation: for integers $\mu, v \geqslant 0$,

$$\begin{aligned} & c(q, \mu, v)\, x^\mu y^v F(q^{\mu+v} x, q^{\mu+v} y) \\ &\quad = F(x, y) - \left(\sum_{i=0}^{\mu-1} \sum_{j=0}^{\infty} + \sum_{i=0}^{\infty} \sum_{j=0}^{v-1} - \sum_{i=0}^{\mu-1} \sum_{j=0}^{v-1} \right) c_{ij} x^i y^j, \end{aligned} \tag{2.66}$$

where $c(q, \mu, \nu) \neq 0$ *is a constant depending on* q, μ, *and* ν. *Let*

$$I(x, y) = \frac{1}{2\pi i} \int_\Gamma \frac{x^n y^n F(tx, ty)\, dt}{(\prod_{k=0}^n (t - q^k))\, t^{m+1}}, \tag{2.67}$$

where Γ *is a circular containing* $0, q^0, q^1, ..., q^n$, *and let* $W_k(q)$ *be defined by* (2.7) *and*

$$A(x, y) := \sum_{k=0}^n W_k(q) \sum_{\mu+\nu=k} \alpha_{\mu\nu k}\, c^{-1}(q, \mu, \nu)\, x^{n-\mu} y^{n-\nu}, \tag{2.68}$$

where $\alpha_{\mu\nu k} \in \mathbb{C}$ *are given by*

$$\sum_{\mu+\nu=k} \alpha_{\mu\nu k} = 1, \tag{2.69}$$

and let

$$\begin{aligned} B(x, y) := {} & \frac{x^n y^n}{m!} \frac{d^m}{dt^m} \left\{ \frac{F(tx, ty)}{\prod_{k=0}^n (t - q^k)} \right\}_{t=0} \\ & - \sum_{k=0}^n W_k(q) \sum_{\mu+\nu=k} \alpha_{\mu\nu k}\, c^{-1}(q, \mu, \nu)\, x^{n-\mu} y^{n-\nu} \\ & \times \left(\sum_{i=0}^{\mu-1} \sum_{j=0}^{m+n+\nu} + \sum_{i=0}^{m+n+\mu} \sum_{j=0}^{\nu-1} - \sum_{i=0}^{\mu-1} \sum_{j=0}^{\nu-1} \right) c_{ij} x^i y^j. \end{aligned} \tag{2.70}$$

Then

(i) $$A(x, y)\, F(x, y) + B(x, y) = \sum_{(i,j) \in \mathbb{N}^2 \setminus E} d_{ij} x^i y^j, \qquad d_{ij} \in \mathbb{C}; \tag{2.71}$$

(ii) $$A(x, y) = \sum_{(i,j) \in N} a_{ij} x^i y^j, \qquad a_{ij} \in \mathbb{C}, \tag{2.72}$$

$$B(x, y) = \sum_{(i,j) \in M} b_{ij} x^i y^j, \qquad b_{ij} \in \mathbb{C}; \tag{2.73}$$

(iii) $$M \subseteq E \qquad \textit{and} \qquad \#(E \setminus M) \geqslant \#N - 1, \tag{2.74}$$

and then the (M, N) *non-homogeneous multivariate Padé approximant to* $F(x, y)$ *on the set* E *is*

$$[M/N]_E\, (x, y) = -\frac{B(x, y)}{A(x, y)}. \tag{2.75}$$

The reader may find the presence of the arbitrary constants $\alpha_{\mu\nu k}$ disconcerting. This is evidence of the high degree of non-uniqueness or non-normality of the unreduced forms of the numerators and denominators of multivariate Padé approximants. See Cuyt [2] for a discussion of uniqueness of the reduced form.

EXAMPLE 4. Let $|q| < 1$, and

$$F(x, y) := \sum_{i, j=0}^{\infty} q^{(i+j)(i+j-1)/2} x^i y^j. \tag{2.76}$$

Note that this is the same function as in Example 2, modulo an inversion $q \to 1/q$ and a scaling of the variables x, y. However, we believe the alternative formulation is of some interest.

For $\mu, \nu \geqslant 0$,

$$\begin{aligned} q^{(\mu+\nu)(\mu+\nu-1)/2} & x^\mu y^\nu F(q^{\mu+\nu} x, q^{\mu+\nu} y) \\ &= \sum_{i, j=0}^{\infty} q^{(i+j)(i+j-1)/2 + (\mu+\nu)(i+j) + (\mu+\nu)(\mu+\nu-1)/2} x^{\mu+i} y^{\nu+j} \\ &= \sum_{i, j=0}^{\infty} q^{(i+j+\mu+\nu)(i+j+\mu+\nu-1)/2} x^{\mu+i} y^{\nu+j} \\ &= \sum_{i=\mu}^{\infty} \sum_{j=\nu}^{\infty} q^{(i+j)(i+j-1)/2} x^i y^j \\ &= F(x, y) - \left(\sum_{i=0}^{\mu-1} \sum_{j=0}^{\infty} + \sum_{i=0}^{\infty} \sum_{j=0}^{\nu-1} - \sum_{i=0}^{\mu-1} \sum_{j=0}^{\nu-1} \right) c_{ij} x^i y^j. \end{aligned} \tag{2.77}$$

Then if M, N, E are defined as in Theorem 2.4, we can explicitly construct the (M, N) non-homogeneous multivariate Padé approximant to $F(x, y)$ on the set E, and the denominator of $[M/N]_E(x, y)$ to $F(x, y)$ is

$$\begin{aligned} A(x, y) &= \sum_{k=0}^{n} W_k(q) \sum_{\mu+\nu=k} \alpha_{\mu\nu k} q^{-(\mu+\nu)(\mu+\nu-1)/2} x^{n-\mu} y^{n-\nu} \\ &= \sum_{k=0}^{n} W_k(q)\, q^{-k(k-1)/2} \sum_{\mu+\nu=k} \alpha_{\mu\nu k} x^{n-\mu} y^{n-\nu} \\ &= \frac{1}{(1-q)^n\, [n]!} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{-k(m+n)} \sum_{\mu+\nu=k} \alpha_{\mu\nu k} x^{n-\mu} y^{n-\nu}, \end{aligned} \tag{2.78}$$

by (2.7), where $\alpha_{\mu\nu k}$ is defined by (2.69).

Now if we let

$$A^*(x, y) := q^{2n(m+n)}(1-q)^n\,[n]!\,A(x, y), \tag{2.79}$$

then the denominator of $[M/N]_E(x, y)$ to $F(x, y)$ is

$$\begin{aligned} A^*(x, y) &= q^{2n(m+n)}(1-q)^n\,[n]!\,A(x, y) \\ &= \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{2n(m+n)-k(m+n)} \sum_{\mu+\nu=k} \alpha_{\mu\nu k} x^{n-\mu} y^{n-\nu} \\ &= \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \sum_{\mu+\nu=k} \alpha_{\mu\nu k} q^{(m+n)(2n-\mu-\nu)} x^{n-\mu} y^{n-\nu} \\ &= \sum_{k=0}^{n} (-1)^{2n-k} \begin{bmatrix} n \\ k \end{bmatrix} \sum_{\mu+\nu=k} \alpha_{\mu\nu k} \\ &\quad \times q^{(m+n)(n-\mu)+(m+n)(n-\nu)} x^{n-\mu} y^{n-\nu} \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \sum_{\mu+\nu=k} \alpha_{\mu\nu k} (-q^{m+n}x)^{n-\mu} (-q^{m+n}y)^{n-\nu}. \end{aligned} \tag{2.80}$$

We note that this result is very similar to the result obtained by P. Wynn (see [1, 7]) for the one variable partial theta function:

$$T_q(x) := \sum_{j=0}^{\infty} q^{j(j-1)/2} x^j, \qquad |q| < 1. \tag{2.81}$$

The denominator of the (m, n) Padé approximant to $T_q(x)$ is

$$R_{m,n}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-q^m x)^k \qquad (m \geqslant n-1 \geqslant 0). \tag{2.82}$$

Lubinsky and Saff have investigated the distribution of the zeros of $R_{m,n}$, the Rogers–Szegö polynomials, in detail (see [7]) and proved some surprising convergence results to $T_q(x)$. We hope to provide a similar analysis for the multivariate partial theta function along the lines of the results of Driver [5, 6].

3. PROOFS

Proof of Theorem 2.1. (i) From the functional equation (2.5), we have for $k \geqslant 1$,

$$
\begin{aligned}
F(q^k x, q^k y) &= F(q \cdot q^{k-1}x, q \cdot q^{k-1}y) \\
&= R_1(q^{k-1}x, q^{k-1}y)\, F(q^{k-1}x, q^{k-1}y) + R_0(q^{k-1}x, q^{k-1}y) \\
&= R_1(q^{k-1}x, q^{k-1}y)\, R_1(q^{k-2}x, q^{k-2}y)\, F(q^{k-2}x, q^{k-2}y) \\
&\quad + R_1(q^{k-1}x, q^{k-1}y)\, R_0(q^{k-2}x, q^{k-2}y) + R_0(q^{k-1}x, q^{k-1}y) \\
&= \cdots \\
&= \left(\prod_{j=0}^{k-1} R_1(q^j x, q^j y) \right) F(x, y) \\
&\quad + \sum_{j=0}^{k-1} R_0(q^j x, q^j y) \prod_{i=j+1}^{k-1} R_1(q^i x, q^i y). \qquad (3.1)
\end{aligned}
$$

Now by the residue theorem and (3.1),

$$
\begin{aligned}
I(x, y) &= \frac{1}{2\pi i} \int_\Gamma \frac{F(tx, ty)\, dt}{\left(\prod_{k=0}^n (t - q^k)\right) t^{m+1}} \\
&= \sum_{k=0}^n W_k(q)\, F(q^k x, q^k y) + \frac{1}{m!} \frac{d^m}{dt^m} \left\{ \frac{F(tx, ty)}{\prod_{k=0}^n (t - q^k)} \right\}_{t=0} \\
&= \sum_{k=0}^n W_k(q) \left(\prod_{j=0}^{k-1} R_1(q^j x, q^j y) \right) F(x, y) \\
&\quad + \sum_{k=0}^n W_k(q) \sum_{j=0}^{k-1} R_0(q^j x, q^j y) \prod_{i=j+1}^{k-1} R_1(q^i x, q^i y) \\
&\quad + \frac{1}{m!} \frac{d^m}{dt^m} \left\{ \frac{F(tx, ty)}{\prod_{k=0}^n (t - q^k)} \right\}_{t=0} \\
&= A(x, y)\, F(x, y) + B(x, y).
\end{aligned}
$$

This completes the proof of (i).

(ii) As $F(x, y)$ is entire in $\mathbb{C} \times \mathbb{C}$, we have the Taylor expansion of $F(x, y)$:

$$
F(x, y) = \sum_{(i, j) \in \mathbb{N}^2} c_{ij} x^i y^j, \qquad c_{ij} \in \mathbb{C}. \tag{3.2}
$$

So

$$
F(tx, ty) = \sum_{(i, j) \in \mathbb{N}^2} c_{ij} t^{i+j} x^i y^j. \tag{3.3}
$$

From (2.6) we observe that the denominator in $I(x, y)$ is a polynomial of degree $m + n + 2$ in t, and then any terms of $F(tx, ty)$ of order less than

$m+n+1$ in t vanish on integration, so (2.11) holds. This completes the proof of (ii).

(iii) As $R_1(x, y)$ is a polynomial of degree 1 in x and y, then $\prod_{j=0}^{k-1} R_1(q^j x, q^j y)$ is a polynomial of degree k in x and y. Thus $A(x, y)$ is a polynomial of degree n in x and y, i.e., (2.12) holds. Now we observe that

$$\frac{d^m}{dt^m}(t^l)\,|_{t=0} = 0, \qquad l > m;$$

then

$$\frac{d^m}{dt^m}\left\{\frac{F(tx, ty)}{\prod_{k=0}^{n}(t-q^k)}\right\}_{t=0} = \sum_{(i,\,j)\in W} b'_{ij}x^i y^j, \qquad b'_{ij} \in \mathbb{C}. \tag{3.4}$$

Also

$$\sum_{k=0}^{n} W_k(q) \sum_{j=0}^{k-1} R_0(q^j x, q^j y) \prod_{i=j+1}^{k-1} R_1(q^i x, q^i y)$$
$$= \sum_{(i,\,j)\in N} b''_{ij}x^i y^j, \qquad b''_{ij} \in \mathbb{C}. \tag{3.5}$$

Then $B(x, y)$ is a polynomial on the set M, i.e., (2.13) holds.

(iv) See Fig. 5 for $n < m < 2n$ and Fig. 6 for $m \geqslant 2n$.

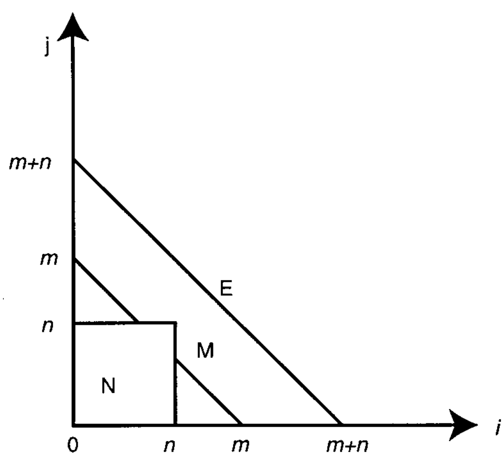

Fig. 5. Index sets M, N and interpolation set E (for $n < m < 2n$).

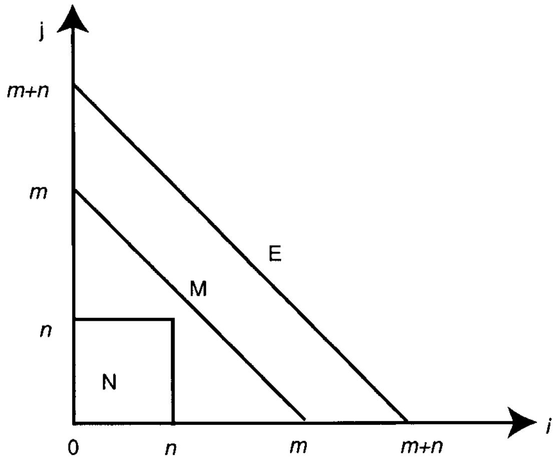

Fig. 6. Index sets M, N and interpolation set E (for $m > 2n$).

Now $M \subseteq E$ is obvious, and

$$\# W = \# \{(i, j) \colon 0 \leqslant i + j \leqslant m,\ i, j \geqslant 0\}$$

$$= \frac{(m+1)(m+2)}{2}.$$

For $n < m < 2n$, i.e., $n + 1 \leqslant m \leqslant 2n - 1$ (see Fig. 5),

$$m - n \geqslant 1 \qquad \text{and} \qquad 2n - m \geqslant 1, \tag{3.6}$$

and

$$\# M = \# N + 2 \cdot \frac{(m-n)(m-n+1)}{2}$$

$$= (n+1)^2 + (m-n)(m-n+1)$$

$$= m^2 + 2n^2 - 2mn + m + n + 1,$$

$$\# E = \frac{(m+n+1)(m+n+2)}{2}$$

so

$$\begin{aligned}
\#(E\backslash M) &= \tfrac{1}{2}(m+n+1)(m+n+2)-(m^2+2n^2-2mn+m+n+1)\\
&= 3mn-\tfrac{1}{2}m^2-\tfrac{3}{2}n^2+\tfrac{1}{2}m+\tfrac{1}{2}n\\
&= mn-\tfrac{1}{2}m(m-n)+\tfrac{3}{2}n(m-n)+\tfrac{1}{2}(m+n)\\
&= mn+\tfrac{1}{2}(m-n)(3n-m)+\tfrac{1}{2}(m+n)\\
&\geqslant mn+\tfrac{1}{2}(n+1)+\tfrac{1}{2}(m+n) \qquad \text{(by (3.6))}\\
&\geqslant (n+1)\,n+\tfrac{1}{2}n+\tfrac{1}{2}(n+n)+\tfrac{1}{2}\\
&\geqslant n^2+2n\\
&= \#N-1.
\end{aligned}$$

Now for $m \geqslant 2n$, we have $N \subset M$, and

$$E\backslash M=\{(i,\ j): m+1 \leqslant i+j \leqslant m+n,\ i,\ j \geqslant 0\};$$

then

$$\begin{aligned}
\#(E\backslash M) &= \frac{(m+n+1)(m+n+2)}{2}-\frac{(m+1)(m+2)}{2}\\
&= \frac{n(2m+n+3)}{2}\\
&\geqslant \frac{n(5n+3)}{2} \qquad (\text{as } m \geqslant 2n)\\
&\geqslant n^2+2n\\
&= \#N-1.
\end{aligned} \tag{3.7}$$

Then for all $m \geqslant n+1$,

$$\#(E\backslash M) \geqslant \#N-1.$$

Combining (i)–(iv), we have

$$[M/N]_E(x,\ y)=-\frac{B(x,\ y)}{A(x,\ y)}.$$

Finally as we let

$$F(x,\ y):=\sum_{i,\ j=0}^{\infty} c_{ij}x^iy^j,$$

and

$$\frac{1}{\prod_{k=0}^{n}(t-q^k)}=(-1)^{n+1}\,q^{-n(n+1)/2}\prod_{k=0}^{n}\left(1-\frac{t}{q^k}\right)^{-1}$$
$$=\frac{(-1)^{n+1}}{q^{n(n+1)/2}}\prod_{k=0}^{n}\left(\sum_{j=0}^{\infty}\left(\frac{t}{q^k}\right)^j\right)$$
$$=\frac{(-1)^{n+1}}{q^{n(n+1)/2}}\sum_{j_0,\,j_1,\,\ldots,\,j_n\geqslant 0} t^{j_0+j_1+\cdots+j_n}q^{-\sum_{k=0}^{n}kj_k},$$

then

$$\frac{F(tx,ty)}{\prod_{k=0}^{n}(t-q^k)}$$
$$=\sum_{r,\,s=0}^{\infty}\frac{(-1)^{n+1}}{q^{n(n+1)/2}}\,c_{rs}t^{r+s}x^ry^s\sum_{j_0,\,j_1,\,\ldots,\,j_n\geqslant 0} t^{j_0+j_1+\cdots+j_n}q^{-\sum_{k=0}^{n}kj_k}$$
$$=\sum_{r,\,s=0}^{\infty}\frac{(-1)^{n+1}}{q^{n(n+1)/2}}\,c_{rs}x^ry^s\sum_{j_0,\,j_1,\,\ldots,\,j_n\geqslant 0} t^{j_0+j_1+\cdots+j_n+r+s}q^{-\sum_{k=0}^{n}kj_k},$$

and

$$\frac{1}{m!}\frac{d^m}{dt^m}\left\{\frac{F(tx,ty)}{\prod_{k=0}^{n}(t-q^k)}\right\}_{t=0}$$
$$=\frac{(-1)^{n+1}}{m!\,q^{n(n+1)/2}}\sum_{\substack{r,\,s,\,j_0,\,\ldots,\,j_n\geqslant 0\\ r+s+j_0+\cdots+j_n=m}} m!\,c_{rs}x^ry^sq^{-\sum_{k=0}^{n}kj_k}$$
$$=\frac{(-1)^{n+1}}{q^{n(n+1)/2}}\sum_{l=0}^{m}\left(\sum_{\substack{j_0,\,\ldots,\,j_n\geqslant o\\ j_0+\cdots+j_n=m-l}} q^{-\sum_{k=0}^{n}kj_k}\right)\sum_{\substack{r,\,s\geqslant 0\\ r+s=l}} c_{rs}x^ry^s,$$

and then

$$B(x,y):=\sum_{k=0}^{n}W_k(q)\sum_{j=0}^{k-1}R_0(q^jx,q^jy)\prod_{i=j+1}^{k-1}R_1(q^ix,q^iy)$$
$$+(-1)^{n+1}\,q^{-n(n+1)/2}\sum_{l=0}^{m}\left(\sum_{\substack{j_0,\,\ldots,\,j_n\geqslant o\\ j_0+\cdots+j_n=m-l}} q^{-\sum_{k=0}^{n}kj_k}\right)\sum_{\substack{r,\,s\geqslant 0\\ r+s=l}} c_{rs}x^ry^s.$$

This completes the proof of Theorem 2.1. ∎

Proof of Theorem 2.2. The assertions (2.21) and (2.22) follow by setting $x=0$ and $y=0$ in (2.20), respectively. The assertions (2.23) and (2.24) easily follow by induction of the functional equations (2.21) and (2.22). Now we prove (2.25).

If we let $R_{1,1}$, S_1, T_1, $R_{0,1}$ be defined by (2.26) to (2.29), then (2.20) directly shows that (2.25) holds for $k=1$. From (2.20)–(2.22)

$$\begin{aligned}F(q^2x, q^2y) &= F(q\cdot qx, q\cdot qy)\\ &= R_1(qx, qy)[R_1(x, y) F(x, y) + u(x) F(x, 0)\\ &\quad + v(y) F(0, y) + R_0(x, y)] + u(qx)(u_1(x) F(x, 0) + u_0(x))\\ &\quad + v(qy)(v_1(y) F(0, y) + v_0(y)) + R_0(qx, qy)\\ &= R_1(qx, qy) R_1(x, y) F(x, y)\\ &\quad + [R_1(qx, qy) u(x) + u(qx) u_1(x)] F(x, 0)\\ &\quad + [R_1(qx, qy) v(y) + v(qy) v_1(y)] F(0, y)\\ &\quad + [R_0(qx, qy) + R_0(x, y) R_1(qx, qy)\\ &\quad + u(qx) u_0(x) + v(qy) v_0(y)]\\ &= R_{1,2}(x, y) F(x, y) + S_2(x, y) F(x, 0)\\ &\quad + T_2(x, y) F(0, y) + R_{0,2}(x, y);\end{aligned}$$

then (2.25) holds for $k=2$. Now we assume that (2.25) holds for $k=n\geqslant 1$, i.e.,

$$\begin{aligned}F(q^nx, q^ny) = R_{1,n}(x, y) F(x, y) + S_n(x, y) F(x, 0)\\ + T_n(x, y) F(0, y) + R_{0,n}(x, y),\end{aligned} \tag{3.8}$$

where $R_{1,n}(x, y)$, $S_n(x, y)$, $T_n(x, y)$, and $R_{0,n}(x, y)$ are defined by (2.26) to (2.29). Then by (2.20), (2.23), and (2.24), for $k=n+1$,

$$\begin{aligned}F(q^{n+1}x, q^{n+1}y) &= F(q\cdot q^nx, q\cdot q^ny)\\ &= R_1(q^nx, q^ny) F(q^nx, q^ny) + u(q^nx) F(q^nx, 0)\\ &\quad + v(q^ny) F(0, q^ny) + R_0(q^nx, q^ny)\\ &= R_1(q^nx, q^ny)\{R_{1,n}(x, y) F(x, y)\\ &\quad + S_n(x, y) F(x, 0) + T_n(x, y) F(0, y)\\ &\quad + R_{0,n}(x, y)\} + u(q^nx)[u_{1,n}(x) F(x, 0) + u_{0,n}(x)]\\ &\quad + v(q^ny)[v_{1,n}(y) F(0, y) + v_{0,n}(y)] + R_0(q^nx, q^ny)\end{aligned}$$

$$= R_1(q^n x, q^n y)\, R_{1,n}(x, y)\, F(x, y) + [R_1(q^n x, q^n y)\, S_n(x, y) + u(q^n x)\, u_{1,n}(x)]\, F(x, 0) + [R_1(q^n x, q^n y)\, T_n(x, y) + v(q^n y)\, v_{1,n}(y)]\, F(0, y) + [R_1(q^n x, q^n y)\, R_{0,n}(x, y) + u(q^n x)\, u_{0,n}(x) + v(q^n y)\, v_{0,n}(y) + R_0(q^n x, q^n y)]. \quad (3.9)$$

Now

$$R_1(q^n x, q^n y)\, R_{1,n}(x, y) = R_{1,n+1}(x, y),$$

$$\begin{aligned} &R_1(q^n x, q^n y)\, S_n(x, y) + u(q^n x)\, u_{1,n}(x) \\ &\quad = R_1(q^n x, q^n y) \sum_{j=0}^{n-1} u(q^j x)\, u_{1,j}(x) \prod_{i=j+1}^{n-1} R_1(q^i x, q^i y) + u(q^n x)\, u_{1,n}(x) \\ &\quad = \sum_{j=0}^{n-1} u(q^j x)\, u_{1,j}(x) \prod_{i=j+1}^{n} R_1(q^i x, q^i y) + u(q^n x)\, u_{1,n}(x) \\ &\quad = \sum_{j=0}^{n} u(q^j x)\, u_{1,j}(x) \prod_{i=j+1}^{n} R_1(q^i x, q^i y) \\ &\quad = S_{n+1}(x, y); \end{aligned}$$

similarly,

$$R_1(q^n x, q^n y)\, T_n(x, y) + v(q^n y)\, v_{1,n}(y) = T_{n+1}(x, y),$$

and

$$\begin{aligned} &R_1(q^n x, q^n y)\, R_{0,n}(x, y) + u(q^n x)\, u_{0,n}(x) + v(q^n y)\, v_{0,n}(y) + R_0(q^n x, q^n y) \\ &\quad = R_1(q^n x, q^n y) \sum_{j=0}^{n-1} [R_0(q^j x, q^j y) + u(q^j x)\, u_{0,j}(x) \\ &\qquad + v(q^j y)\, v_{0,j}(y)] \prod_{i=j+1}^{n-1} R_1(q^i x, q^i y) + u(q^n x)\, u_{0,n}(x) \\ &\qquad + v(q^n y)\, v_{0,n}(y) + R_0(q^n x, q^n y) \\ &\quad = \sum_{j=0}^{n-1} [R_0(q^j x, q^j y) + u(q^j x)\, u_{0,j}(x) + v(q^j y)\, v_{0,j}(y)] \prod_{i=j+1}^{n} R_1(q^i x, q^i y) \\ &\qquad + u(q^n x)\, u_{0,n}(x) + v(q^n y)\, v_{0,n}(y) + R_0(q^n x, q^n y) \\ &\quad = \sum_{j=0}^{n} [R_0(q^j x, q^j y) + u(q^j x)\, u_{0,j}(x) + v(q^j y)\, v_{0,j}(y)] \prod_{i=j+1}^{n} R_1(q^i x, q^i y) \\ &\quad = R_{0,n+1}(x, y). \end{aligned}$$

Putting above three equations into (3.9), we have

$$F(q^{n+1}x, q^{n+1}y) = R_{1,n+1}(x, y)\, F(x, y) + S_{n+1}(x, y)\, F(x, 0) + T_{n+1}(x, y)\, F(0, y) + R_{0,n+1}(x, y), \tag{3.10}$$

i.e., (2.25) holds for $k = n+1$. By mathematical induction, we have that (2.25) holds for all integers $k \geqslant 1$.

This completes the proof of Theorem 2.2. ▮

Proof of Theorem 2.3. (i) By the residue theorem and Theorem 2.2,

$$\begin{aligned} I(x, y) &= \frac{1}{2\pi i} \int_\Gamma \frac{F(tx, ty)\, dt}{\left(\prod_{k=0}^{n} (t - q^k)\right) t^{m+2n+1}} \\ &= \sum_{k=0}^{n} \frac{F(q^k x, q^k y)}{\left(\prod_{h=0, h\neq k}^{n} (q^k - q^h)\right) q^{k(m+2n+1)}} \\ &\quad + \frac{1}{(m+2n)!} \frac{d^{m+2n}}{dt^{m+2n}} \left\{ \frac{F(tx, ty)}{\prod_{k=0}^{n} (t - q^k)} \right\}_{t=0} \\ &= \sum_{k=0}^{n} W'_k(q) [R_{1,k}(x, y)\, F(x, y) + S_k(x, y)\, F(x, 0) \\ &\quad + T_k(x, y)\, F(0, y) + R_{0,k}(x, y)] \\ &\quad + \frac{1}{(m+2n)!} \frac{d^{m+2n}}{dt^{m+2n}} \left\{ \frac{F(tx, ty)}{\prod_{k=0}^{n} (t - q^k)} \right\}_{t=0} \\ &= R(x, y)\, F(x, y) + S(x, y)\, F(x, 0) + T(x, y)\, F(0, y) + D(x, y). \end{aligned}$$

Parts (ii) and (iii) are obvious.

(iv) From (i) and (iii),

$$\begin{aligned} I(x, y) &= R(x, y)\, F(x, y) + S(x, y)\, F(x, 0) + T(x, y)\, F(0, y) + D(x, y) \\ &= R(x, y)\, F(x, y) + \sum_{i,j=0}^{n} s_{ij} x^i y^j F(x, 0) \\ &\quad + \sum_{i,j=0}^{n} t_{ij} x^i y^j F(0, y) + D(x, y) \\ &= R(x, y)\, F(x, y) + B(x, y) + \sum_{i,j=0}^{n} s_{ij} x^i y^j \sum_{k=m+2n-i+1}^{\infty} c_{k0} x^k \\ &\quad + \sum_{i,j=0}^{n} t_{ij} x^i y^j \sum_{l=m+2n-j+1}^{\infty} c_{0l} y^l, \end{aligned} \tag{3.11}$$

where $B(x, y)$ is defined by (2.48). Now as $D(x, y)$ is a polynomial on the set W, and

$$\sum_{i,\, j=0}^{n} s_{ij} x^i y^j F_{m+2n-i}(x, 0) = \sum_{i,\, j=0}^{n} s_{ij} x^i y^j \sum_{k=0}^{m+2n-i} c_{k0} x^k$$

$$= \sum_{i,\, j=0}^{n} s_{ij} y^j \sum_{k=0}^{m+2n-i} c_{k0} x^{k+i}$$

$$=: \sum_{(i,\, j) \in U} u_{ij} x^i y^j, \qquad u_{ij} \in \mathbb{C}, \tag{3.12}$$

and similarly

$$\sum_{i,\, j=0}^{n} t_{ij} x^i y^j F_{m+2n-j}(0, y) = \sum_{(i,\, j) \in V} v_{ij} x^i y^j, \qquad v_{ij} \in \mathbb{C}, \tag{3.13}$$

then $B(x, y)$ is a polynomial on the set M, i.e., (2.49) holds. Now in (3.8),

$$\sum_{i,\, j=0}^{n} s_{ij} x^i y^j \sum_{k=m+2n-i+1}^{\infty} c_{k0} x^k = \sum_{i,\, j=0}^{n} s_{ij} y^j \sum_{k=m+2n+1}^{\infty} c_{k-i,\, 0} x^k$$

$$=: \sum_{i=m+2n+1}^{\infty} \sum_{j=0}^{n} g_{ij} x^i y^j, \qquad g_{ij} \in \mathbb{C}, \tag{3.14}$$

$$\sum_{i,\, j=0}^{n} t_{ij} x^i y^j \sum_{l=m+2n-j+1}^{\infty} c_{0l} y^l = \sum_{i,\, j=0}^{n} t_{ij} x^i \sum_{l=m+2n+1}^{\infty} c_{0,\, l-j} y^l$$

$$=: \sum_{i=0}^{n} \sum_{j=m+2n+1}^{\infty} h_{ij} x^i y^j, \qquad h_{ij} \in \mathbb{C}, \tag{3.15}$$

and from (ii), $I(x, y)$ involves only term $x^i y^i$ with $(i, j) \in \mathbb{N}^2 \backslash E$; then we have

$$R(x, y)\, F(x, y) + B(x, y) = \sum_{(i,\, j) \in \mathbb{N}^2 \backslash E} e_{ij} x^i y^j, \qquad e_{ij} \in \mathbb{C}.$$

This completes the proof of (iv).

(v) $M \subseteq E$ is obvious (see Figs. 1 to 4), and

$$E \backslash M = \{(i, j) \colon i, j \geqslant n+1,\, m+2n < i+j \leqslant m+3n\}$$

$$= \{(i, j) \colon i, j \geqslant n+1,\, m+2n+1 \leqslant i+j \leqslant m+3n\},$$

so

$$\begin{aligned}
\#(E\backslash M) &= \#\{(i, j)\colon i, j \geqslant n+1, m+2n+1 \leqslant i+j \leqslant m+3n\} \\
&= \#\{(i, j)\colon i, j \geqslant 1, m+1 \leqslant i+j \leqslant m+n\} \\
&= \tfrac{1}{2}(m+n)(m+n+1) - \tfrac{1}{2}m(m+1) \\
&= \tfrac{1}{2}n(2m+n+1) \\
&\geqslant n^2+2n \\
&= \#N-1,
\end{aligned}$$

provided

$$\frac{1}{2}n(2m+n+1)-(n^2+2n) \geqslant 0 \Leftrightarrow m \geqslant \frac{n+3}{2}. \tag{3.16}$$

As we assumed this, we have the result.

Combining (i)–(iv), we have for $F(x, y)$,

$$[M/N]_E(x, y) = -\frac{B(x, y)}{R(x, y)}.$$

Similar to the proof of (2.15) in Theorem 2.1, we have (2.52).

This completes the proof of Theorem 2.3. ▮

Proof of Theorem 2.4. (i) By the residue theorem and the functional equation (2.66),

$$\begin{aligned}
I(x, y) &= \sum_{k=0}^{n} \frac{x^n y^n F(q^k x, q^k y)}{(\prod_{h=0, h\neq k}^{n} (q^k - q^h))\, q^{k(m+1)}} + \frac{x^n y^n}{m!} \frac{d^m}{dt^m} \left\{ \frac{F(tx, ty)}{\prod_{k=0}^{n} (t-q^k)} \right\}_{t=0} \\
&= \sum_{k=0}^{n} W_k(q) \sum_{\mu+\nu=k} \alpha_{\mu\nu k} x^{n-\mu} y^{n-\nu} x^{\mu} y^{\nu} F(q^{\mu+\nu} x, q^{\mu+\nu} y) \\
&\quad + \frac{x^n y^n}{m!} \frac{d^m}{dt^m} \left\{ \frac{F(tx, ty)}{\prod_{k=0}^{n} (t-q^k)} \right\}_{t=0} \\
&= \sum_{k=0}^{n} W_k(q) \sum_{\mu+\nu=k} \alpha_{\mu\nu k} c^{-1}(q, \mu, \nu)\, x^{n-\mu} y^{n-\nu} \\
&\quad \times \left[F(x, y) - \left(\sum_{i=0}^{\mu-1} \sum_{j=0}^{\infty} + \sum_{i=0}^{\infty} \sum_{j=0}^{\nu-1} - \sum_{i=0}^{\mu-1} \sum_{j=0}^{\nu-1} \right) c_{ij} x^i y^j \right] \\
&\quad + \frac{x^n y^n}{m!} \frac{d^m}{dt^m} \left\{ \frac{F(tx, ty)}{\prod_{k=0}^{n} (t-q^k)} \right\}_{t=0}
\end{aligned}$$

$$= \sum_{k=0}^{n} W_k(q) \sum_{\mu+\nu=k} \alpha_{\mu\nu k} c^{-1}(q, \mu, \nu)\, x^{n-\mu} y^{n-\nu} F(x, y)$$

$$- \sum_{k=0}^{n} W_k(q) \sum_{\mu+\nu=k} \alpha_{\mu\nu k} c^{-1}(q, \mu, \nu)\, x^{n-\mu} y^{n-\nu}$$

$$\times \left(\sum_{i=0}^{\mu-1} \sum_{j=0}^{m+n+\nu} + \sum_{i=0}^{m+n+\mu} \sum_{j=0}^{\nu-1} - \sum_{i=0}^{\mu-1} \sum_{j=0}^{\nu-1} \right) c_{ij} x^i y^j$$

$$- \sum_{k=0}^{n} W_k(q) \sum_{\mu+\nu=k} \alpha_{\mu\nu k} c^{-1}(q, \mu, \nu)\, x^{n-\mu} y^{n-\nu}$$

$$\times \left(\sum_{i=0}^{\mu-1} \sum_{j=m+n+\nu+1}^{\infty} + \sum_{i=m+n+\mu+1}^{\infty} \sum_{j=0}^{\nu-1} \right) c_{ij} x^i y^j$$

$$+ \frac{x^n y^n}{m!} \frac{d^m}{dt^m} \left\{ \frac{F(tx, ty)}{\prod_{k=0}^{n} (t - q^k)} \right\}_{t=0}$$

$$= A(x, y)\, F(x, y) + B(x, y) - D(x, y), \tag{3.17}$$

where

$$D(x, y) := \sum_{k=0}^{n} W_k(q) \sum_{\mu+\nu=k} \alpha_{\mu\nu k} c^{-1}(q, \mu, \nu)\, x^{n-\mu} y^{n-\nu}$$

$$\times \left(\sum_{i=0}^{\mu-1} \sum_{j=m+n+\nu+1}^{\infty} + \sum_{i=m+n+\mu+1}^{\infty} \sum_{j=0}^{\nu-1} - \sum_{i=0}^{\mu-1} \sum_{j=0}^{\nu-1} \right) c_{ij} x^i y^j, \tag{3.18}$$

so

$$A(x, y)\, F(x, y) + B(x, y) = I(x, y) + D(x, y). \tag{3.19}$$

From (2.67) we observe that the denominator in $I(x, y)$ is a polynomial of degree $m+n+2$. Then any terms of $F(tx, ty)$ of orderless than $m+n+1$ in t vanish on integration, so

$$I(x, y) = x^n y^n \sum_{i+j \geqslant m+n+1} e_{ij} x^i y^j, \qquad e_{ij} \in \mathbb{C}$$

$$=: \sum_{(i, j) \in I} e^*_{ij} x^i y^j, \qquad e^*_{ij} \in \mathbb{C}, \tag{3.20}$$

where

$$I := \{(i, j) \colon i, j \geqslant n,\ i + j \geqslant m + 3n + 1\}, \tag{3.21}$$

and

$$I \subset \mathbb{N}^2 \backslash E. \tag{3.22}$$

Now from (3.18), we have

$$D(x, y) = \sum_{(i, j) \in D} h_{ij} x^i y^j, \qquad h_{ij} \in \mathbb{C}, \tag{3.23}$$

where

$$D := \{(i, j): 0 \leqslant i \leqslant n, j \geqslant m + 2n + 1\} \\ \cup \{(i, j): i \geqslant m + 2n + 1, 0 \leqslant j \leqslant n\}, \tag{3.24}$$

and

$$D \subset \mathbb{N}^2 \backslash E. \tag{3.25}$$

Combining (3.19), (3.20), and (3.23), we have (2.71).

This completes the proof of (i).

(ii) (2.72) is obvious. We know that

$$\frac{d^m}{dt^m} \left\{ \frac{F(tx, ty)}{\prod_{k=0}^{n} (t - q^k)} \right\}_{t=0} = \sum_{0 \leqslant i+j \leqslant m} q_{ij} x^i y^j, \qquad g_{ij} \in \mathbb{C}; \tag{3.26}$$

then

$$\frac{x^n y^n}{m!} \frac{d^m}{dt^m} \left\{ \frac{F(tx, ty)}{\prod_{k=0}^{n} (t - q^k)} \right\}_{t=0} = \sum_{(i, j) \in W} g_{ij}^* x^i y^j, \qquad g_{ij}^* \in \mathbb{C}. \tag{3.27}$$

Now

$$\sum_{k=0}^{n} W_k(q) \sum_{\mu + v = k} \alpha_{\mu v k} c^{-1}(q, \mu, v) \, x^{n-\mu} y^{n-v} \sum_{i=0}^{\mu-1} \sum_{j=0}^{m+n+v} c_{ij} x^i y^j \\ = \sum_{(i, j) \in V} v_{ij} x^i y^j, \qquad v_{ij} \in \mathbb{C}, \tag{3.28}$$

$$\sum_{k=0}^{n} W_k(q) \sum_{\mu + v = k} \alpha_{\mu v k} c^{-1}(q, \mu, v) \, x^{n-\mu} y^{n-v} \sum_{i=0}^{m+n+\mu} \sum_{j=0}^{v-1} c_{ij} x^i y^j \\ = \sum_{(i, j) \in U} u_{ij} x^i y^j, \qquad u_{ij} \in \mathbb{C}, \tag{3.29}$$

$$\sum_{k=0}^{n} W_k(q) \sum_{\mu + v = k} \alpha_{\mu v k} c^{-1}(q, \mu, v) \, x^{n-\mu} y^{n-v} \sum_{i=0}^{\mu-1} \sum_{j=0}^{v-1} c_{ij} x^i y^j \\ = \sum_{(i, j) \in N} n_{ij} x^i y^j, \qquad n_{ij} \in \mathbb{C}. \tag{3.30}$$

Combining (2.34), (2.70), and (3.27)–(3.30), we have (2.73).

This completes the proof of (ii).

Part (iii) is the same as (v) in Theorem 2.3.

This completes the proof of Theorem 2.4. ∎

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